Practice Questions for the final exam

Groups

1. (a) Classify finite groups of order 9.
   (b) Classify finite groups of order 45.
2. Prove that a group of order 12 is not a simple group.
3. Let S_3 be the symmetric group of three elements.
   (a) List all the subgroups of S_3.
   (b) List all the normal subgroups of S_3.
4. Assume finite group G acts transitively on set S. Let $x \in G$ and G_x be the stabilizer of x.
   (a) State the counting formula relating $|G|, |S|, |G_x|$.
   (b) Let G be the rotational symmetry group of tetrahedra. Use the counting formula to find $|G|$.

Rings and Fields

1. Find the units in the following rings
   (a) $\mathbb{Z}/12\mathbb{Z}$,
   (b) $\mathbb{Z}[\sqrt{-5}]$,
   (c) $\mathbb{Z}[i]/(i+3)$.
2. Determine whether the following rings are fields and prove your statement for each case.
   (a) $\mathbb{Z}[i]/(2+3i)$,
   (b) $\mathbb{Z}[\sqrt{-2}]/(7)$,
   (c) $\mathbb{Q}[x]/(x^5+3x^3+21x+6)$,
   (d) $\mathbb{Q}[x]/(x^4+x+1)$.
3. Find all the irreducible polynomials of degree 4 in $\mathbb{F}_2[x]$.

4. Let $R = \mathbb{Z}[\omega]$, where $\omega = e^{\frac{2\pi i}{3}}$. Prove that R is isomorphic to $\mathbb{Z}[x]/[x^2 + 3x + 3]$,
5. Let $R = \mathbb{Z}[\omega]$, where $\omega = e^{\frac{2\pi i}{3}}$. Prove that R is a principal ideal domain.
6. Find the degree of the following field extensions
   (a) $\mathbb{Q}(\omega, \sqrt{2})/\mathbb{Q}$, where $\omega = e^{\frac{2\pi i}{3}}$.
   (b) $\mathbb{Q}(\sqrt[6]{6})/\mathbb{Q}$
   (c) $\mathbb{Q}(\sqrt[6]{6})/\mathbb{Q}(\sqrt[2]{6})$
7. Find the irreducible polynomials of α over F
   (a) $\alpha = \sqrt{2} + \sqrt{3}$, $F = \mathbb{Q}$
   (b) $\alpha = \sqrt{2 + \sqrt{2}}$, $F = \mathbb{Q}$
8. Find the degree of the splitting field of the following polynomials $f(x)$ over F.
   (a) $f(x) = (x^2 - 2)(x^2 - 3)$, $F = \mathbb{Q}$
   (b) $f(x) = x^5 - 2$, $F = \mathbb{Q}$
   (c) $f(x) = x^5 - 2$, $F = \mathbb{Q}(\sqrt[5]{2})$
9. Find the Galois groups of the fields extensions
   (a) $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$,
   (b) The splitting field of $f(x) = (x^2 - 2)(x^2 - 3)$ over $\mathbb{Q}$
10. Write the splitting field of $f(x) = (x^2 - 2)(x^2 - 3)$ over $\mathbb{Q}$ in the form $\mathbb{Q}[\alpha]$. (Find the primitive element α)
11. Determine the intermediate fields L in $\mathbb{Q} \subset L \subset \mathbb{Q}(\sqrt{2}, \sqrt{3})$.
12. Find all the constants $a \in \mathbb{F}_3$ such that $\mathbb{F}_3[x]/(x^3 + x^2 + a)$ is field.

1 List of concepts

Groups Subgroups, normal subgroups, group operations, stabilizer, counting formulas, Sylow theorems.

Rings Units, ideal, correspondence theorem, integral domain, maximal ideals, Hilbert's Nullstellensatz, Euclidean domain, PID, UFD, Gauss's lemma, Eisenstein criterion(Cyclotomic polynomials, theorem 12.4.9), irreducibility criterion of polynomials by reduction mod p, Gaussian integers $\mathbb{Z}[i]$, prime elements in $F[x]$, $\mathbb{Z}[i]$.

Fields degree of fields extension, degree of algebraic elements, multiplicative property of degrees, irreducible polynomial of algebraic elements, constructible points by ruler and compass (theorem 15.5.6), primitive element theorem, splitting fields of a polynomial, Galois correspondence.